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The unsteady solutions of a unified heat conduction equation

CHAUR-KIE LIN, CHI-CHUAN HWANG† and YEON-PUN CHANG

Department of Mechanical Engineering, Chung Yuan University, Chung Li, Taiwan 32023, Republic of China

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1. INTRODUCTION

The short-phase laser heating process of metals is composed of three general steps: the deposition of radiation energy on electrons, the transport of energy by electrons and the propagation of energy through media. The propagation of energy during a relatively slow heating process can be modeled by the Fourier heat conduction model, since the deposition of radiation energy can be assumed to be instantaneous. However, it takes time, in reality, to establish an equilibrium state in thermodynamic transition. For a problem involving reflectivity change resulting from short-pulse laser heating on gold films [1], the response time is on the order of picoseconds, comparable to the time required to establish an equilibrium state. The diffusion theory fails under such circumstances because the hot electron gas and the metal lattice cannot reach thermodynamic equilibrium in such a short period of time. Thus, more general and rigorous models are needed to include effects of electron-lattice interactions and non-Fourier transport. After Maxwell's research [2] on the kinetic theory of gases, which has had great influence on the development of the thermal wave theory, modifications on Fourier's law are promoted by its deficiencies in advanced applications [3–16].

The unified model (Tzou [16]) is a generalized approach based on the dual-phase-lag concept which accounts for the lagging behavior in the high-rate response. A universal constitutive equation between the heat flux vector and the temperature gradient is proposed with an effort to cover a wide range of physical responses from microscopic to macroscopic scales in both space and time [16]. An exact solution, using the method of separation of variables, to the above universal constitutive equation for a one-dimensional problem is addressed in this paper. Part of the results are found to be different from those by Tzou [16]. The aim of this note is to present a convenient approach to the short-pulse laser heating problem by virtue of the unified heat conduction equation.

† Author to whom correspondence should be addressed.

2. THEORETICAL ANALYSIS

The short-pulse laser heating of a metal film can be treated as a one-dimensional problem because its heat penetration depth is much smaller than the beam diameter. The solid is assumed to have a finite thickness, l. The phase lag of the heat flux and that of the temperature gradient are τ_q and τ_T, respectively. An initial temperature distribution of constant value, T_0, in solid and an imposed initial time-rate change of temperature, $\dot{T}_0$, are given. A suddenly-raised temperature T_w at left end $x \times 0$ and a zero temperature gradient remaining at right end $x = l$ are suitable boundary conditions for this type of problem. After introducing the following dimensionless variables as in ref. [16],

$$\theta = \frac{T - T_0}{T_w - T_0}, \quad \delta = \frac{x}{l}, \quad \text{and} \quad \beta = \frac{t}{(l^2/\alpha)} \tag{1}$$

the temperature field equation, the initial conditions and the boundary conditions become:

$$\frac{\partial^2 \theta}{\partial \delta^2} + z_T \frac{\partial^3 \theta}{\partial \delta^2 \partial \beta} = \frac{\partial \theta}{\partial \beta} + z_q \frac{\partial^2 \theta}{\partial \beta^2} \tag{2}$$

and

$$\theta = 0 \quad \text{and} \quad \frac{\partial \theta}{\partial \beta} = \dot{\theta}_0 \quad \text{at } \beta = 0 \tag{3}$$

$$\theta = 1 \quad \text{at } \delta = 0 \quad \text{and} \quad \frac{\partial \theta}{\partial \delta} = 0 \quad \text{at } \delta = 1 \tag{4}$$

NOMENCLATURE

A, B	the eigenvalue components of the dimensionless time variable
C	the coefficients for complex conjugate roots
D	the discriminant of the characteristic equation
E, F	the coefficients for distinct real roots
G, H	the coefficients for a double root
l	length of the one-dimensional solid
q	heat flux
t	time variable
T	absolute temperature
x	space variable
z	dimensionless relaxation time.

Greek symbols

α	thermal diffusivity
β	dimensionless time variable
Γ	dimensionless temperature component
δ	dimensionless space variable
λ	the eigenvalue of the dimensionless time variable
μ	the eigenvalue of the dimensionless space variable
τ	phase lag or relaxation time
θ	dimensionless temperature.

Superscripts

+	solutions for distinct real roots
0	solutions for a double root
−	solutions for complex conjugate roots.

Subscripts

n	nth value
0	initial value
q	heat flux vector
T	temperature gradient
w	quantity at the wall.

where

$$z_T = \frac{\tau_T}{(l^2/\alpha)}, \quad z_q = \frac{\tau_q}{(l^2/\alpha)}, \quad \text{and} \quad \dot{\theta}_0 = \dot{T}_0 \frac{l^2/\alpha}{T_w - T_0}$$

By using the method of separation of variables, one can easily find the solutions of equations (2)–(4) as follows:

When $z_q = 0$,

$$\theta_n(\delta,\beta) = 1 + \sum_{n=1}^{\infty} G_n e^{\lambda_n \beta} \sin \mu_n \delta \tag{5}$$

where

$$\mu_n = \frac{2n-1}{2}\pi, \quad \text{for } n = 1, 2, 3, \ldots, \lambda_n = \frac{-\mu_n^2}{1 + z_T \mu_n^2}$$

and

$$G_n = \frac{-2}{\mu_n}.$$

When $z_q \neq 0$,

$$\theta_n(\delta,\beta) = 1 + \sum_{n=1}^{\infty} \Gamma_n \sin \mu_n \delta \tag{6}$$

where

$$\mu_n = \frac{2n-1}{2}\pi, \quad \text{for } n = 1, 2, 3, \ldots,$$

$$\Gamma_n = \begin{cases} \Gamma_n^+ & \text{for } D_n > 0 \\ \Gamma_n^0 & \text{for } D_n = 0 \\ \Gamma_n^- & \text{for } D_n < 0 \end{cases} \tag{7}$$

$$D_n = (1 + z_T \mu_n^2)^2 - 4 z_q \mu_n^2 \tag{8}$$

and

$$\Gamma_n^+(\delta,\beta) = e^{-A_n\beta}[E_n e^{-B_n\beta} + F_n e^{B_n\beta}] \tag{9a}$$

$$\Gamma_n^0(\delta,\beta) = e^{-A_n\beta}[G_n + H_n\beta] \tag{9b}$$

and

$$\Gamma_n(\delta,\beta) = e^{-A_n\beta}[G_n \cos B_n\beta + C_n \sin B_n\beta] \tag{9c}$$

where

$$A_n = \frac{1 + z_T\mu_n^2}{2z_q}, \quad B_n = \frac{\sqrt{|D_n|}}{2z_q},$$

$$E_n = \frac{\dot{\theta}_0 - A_n - B_n}{B_n\mu_n}, \quad F_n = \frac{-\dot{\theta}_0 + A_n - B_n}{B_n\mu_n},$$

$$G_n = \frac{-2}{\mu_n}, \quad H_n = \frac{2(\dot{\theta}_0 - A_n)}{\mu_n}, \quad \text{and} \quad C_n = \frac{H_n}{B_n}.$$

3. RESULTS AND DISCUSSION

Figure 1 shows the temperature distributions along the thickness direction of the metal film. In contrast to Tzou's result [16], several distinct features are found: (1) the temperatures decrease with increasing values of z_T near left end of the film, i.e. $\delta \leqslant 0.2$, which is opposite to the trend in Tzou's; (2) the absolute values of the temperature gradients near left end for $z_T \geqslant 0.04$ are larger than those in Tzou's; (3) unlike $\theta \approx 0.4$ $\delta \approx 1$ for $z_T = 0.5$ in Tzou's, the temperature goes to a small value near right end of the film for the same z_T.

To identify the reason for the above diverging phenomena based upon the same constitutive equation, an attempt is made to simulate the most arguable curve, $z_T = 0.5$, in Fig. 1 of Tzou's [16]; the comparison is shown in Fig. 2. First of all, equation (8) can be rewritten as: $D_n = (1 - z_T\mu_n^2)^2 + 4(z_T - z_q)\mu_n^2$. Thus, D_n is always greater than zero for $z_T > z_q$, which is the case: $z_T = 0.5$ and $z_q = 0.05$ in Fig. 2. Under such circumstances, Γ_n^+, equation (9a), is the only formulation for Γ_n in the summation part of equation (6), for temperature θ. This correct correspondence is depicted as a solid line in Fig. 2. However, when replaced, the above Γ_n^+ by Γ_n^-, equation (9c), into equation (6) for the same case:

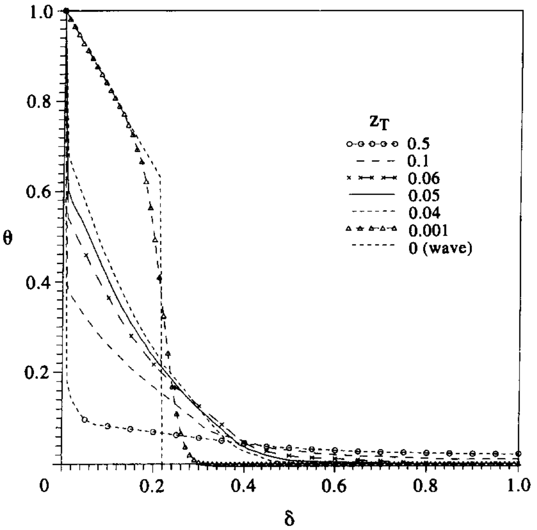

Fig. 1. The temperature distributions along the metal film for $\beta = 0.05$, $z_q = 0.05$, and $z_T = 0.0$ to 0.5.

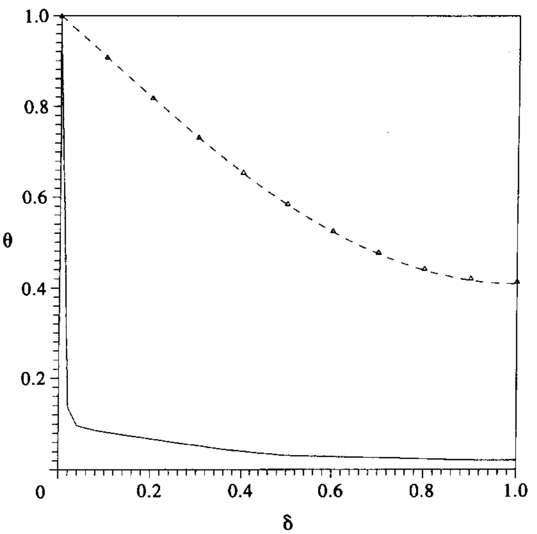

Fig. 2. The temperature distributions along the metal film for $\beta = 0.05$, $z_q = 0.05$, and $z_T = 0.5$. The solid line represents the result using the method of separation of variables, equation (9a). The Δ symbols represent the simulated result using equation (9c) for an inconsistent condition, $D_n > 0$. The dashed line represents the solution of Tzou's study, ref. [16].

$z_T = 0.5$ and $z_q = 0.05$, the calculation is shown using symbol Δ in Fig. 2. A coincident agreement is found between this calculation and that of Tzou's, depicted in dashed line. Because the numerical inversion of Laplace transforms [17] is used, the results in Tzou's [16] missed the condition of positive values of D_n. Thus, it serves to pay attention to the applicability of the method of numerical Laplace inversion, when used, in addition to the choice of parameters for the convergence of the discretization error and the truncation error.

It should also be noted that the solution of θ in equation (6) requires more than one formulation of Γ_n, (Γ_n^+, Γ_n^0, or Γ_n^-), at a fixed position, δ, and at a fixed moment, β, for a fixed set of parameters z_T and z_q (see Appendix).

As for the case when $z_q = z_T$ (not necessarily equal to zero), the temperature distribution with several sets of equal values of z_q and z_T are depicted in Fig. 3. An intrinsic assumption on the classical diffusion theory is an instantaneous response, which means it takes no time for electrons to change their states, or $z_q = z_T = 0$, when establishing an equilibrium state during a thermodynamic transition. However, the lagging behavior of temperature distribution becomes more obvious with increasing values of z_q and z_T even if they are of the same value. Therefore, an equal value of z_q and z_T certainly cannot promise an instantaneously-reached thermodynamic equilibrium state, i.e. a classical diffusion field.

Although a finite value of relaxation time leads to a steep drop of the temperature gradient near left end of the film, see Fig. 3, the temperature distribution tends to return to a 'diffusive' type as time goes, see Fig. 4. The temperature

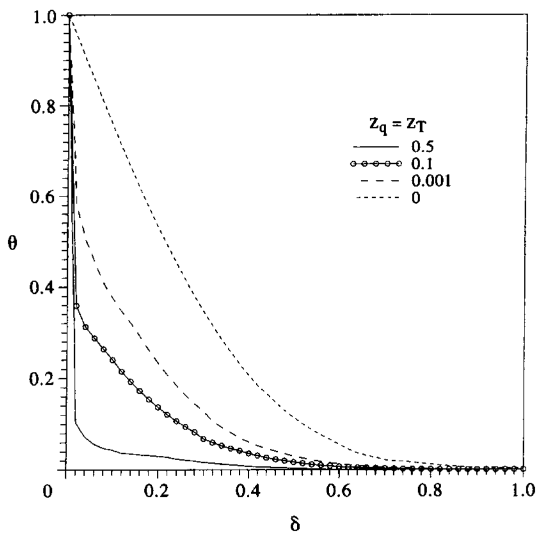

Fig. 3. The temperature distributions along the metal film for $\beta = 0.05$ and $z_q = z_T = 0.0$–0.5.

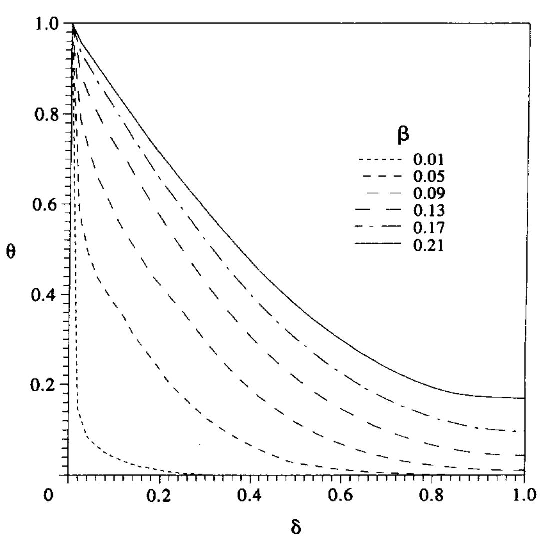

Fig. 4. The temperature distributions along the metal film for $z_q = z_T = 0.05$ and $\beta = 0.01$–0.21.

distribution in the whole film rises to a higher value with increasing time β. The time rate of increment for this temperature evolution near right end of the film is in a speeding-up fashion, while that near the left end is slowing down.

4. CONCLUSION

The method of separation of variables gives a general exact solution to the one-dimensional unified heat conduction equation. This solution is characterized by a summation with term-by-term dependency for a system with fixed intrinsic properties. The reason why a diverging result occurs when using Laplace transform method has been identified. One of the main distinctions falls on the steep drop of the temperature distribution near left end of the film for a finite value of z_T, which demonstrates the delayed response of microstructural effects in space being lumped into the macroscopic lagging behavior. An equal non-zero value for both z_T and z_q leads the system to a lagged temperature distribution rather than a classical diffusion field. However, the steep drop in temperature distribution near the left end of the film due to the lagging effects will be smoothed out into a diffusive type as time elapses.

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APPENDIX

The dependence of D_n on the value of n is listed as follows:
(I) when $z_q = 0$, equation (5) is used for temperature θ;
(II) when $z_q \neq 0$, equation (6) is used for temperature θ, in which

(A) when $z_T = 0$ (classical wave),

$$n < \frac{1}{2} + \frac{1}{2\pi\sqrt{z_q}} \Rightarrow D_n > 0 \tag{1}$$

and equation (9a), Γ_n^+, is used for Γ_n,

$$n = \frac{1}{2} + \frac{1}{2\pi\sqrt{z_q}} \Rightarrow D_n = 0 \tag{2}$$

and equation (9b), Γ_n^0, is used for Γ_n, and

$$n > \frac{1}{2} + \frac{1}{2\pi\sqrt{z_q}} \Rightarrow D_n < 0 \tag{3}$$

and equation (9c), Γ_n^-, is used for Γ_n;

(B) when $z_T \neq 0$,

(1) when $z_T = z_q$,

$$n = \frac{1}{2} + \frac{1}{\pi\sqrt{z_q}} \Rightarrow D_n = 0 \tag{a}$$

and equation (9b), Γ_n^0, is used for Γ_n, and

$$n \neq \frac{1}{2} + \frac{1}{\pi\sqrt{z_q}} \Rightarrow D_n > 0 \tag{b}$$

and equation (9a), Γ_n^+, is used for Γ_n;

(2) when $z_T \neq z_q$,

$$\text{when } z_T > z_q, \quad \Rightarrow D_n > 0 \tag{a}$$

and equation (9a); Γ_n^+, is used for Γ_n;

$$\text{when } z_T < z_q \tag{b}$$

$$n > \frac{1}{2} + \frac{\sqrt{z_q} + \sqrt{z_q - z_T}}{\pi z_T} \quad \text{or} \quad n < \frac{1}{2} + \frac{\sqrt{z_q} - \sqrt{z_q - z_T}}{\pi z_T} \tag{i}$$

$\Rightarrow D_n > 0$ and equation (9a), Γ_n^+, is used for Γ_n,

$$n = \frac{1}{2} + \frac{\sqrt{z_q} \pm \sqrt{z_q - z_T}}{\pi z_T} \tag{ii}$$

$\Rightarrow D_n = 0$ and equation (9b), Γ_n^0, is used for Γ_n, and

$$\frac{1}{2} + \frac{\sqrt{z_q} - \sqrt{z_q - z_T}}{\pi z_T} < n < \frac{1}{2} + \frac{\sqrt{z_q} + \sqrt{z_q - z_T}}{\pi z_T} \tag{iii}$$

$\Rightarrow D_n < 0$ and equation (9c), Γ_n^-, is used for Γ_n.